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CORRECTION TO "COMPLETE SURFACES OF FINITE TOTAL CURVATURE"

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Let S be a compact subset of a smooth complete two-dimensional riemannian manifold M. Let

$$\Omega(r) = \{x \in M\colon \operatorname{dist}(x, S) < r\}, \qquad \Gamma(r) = \partial\Omega(r),$$

and let $L(r)$ be the length of $\Gamma(r)$, [2, equation (2), p. 317] gives a formula for the $L'(r)$. Peter Li has noticed that the formula does not always hold. However, the left-hand side of the equation is always less than or equal to the right-hand side, and the inequality suffices for the applications in the rest of the paper. The correct formula (which implies the inequality) is as follows.

Proposition. *If $\Gamma(r)$ is a piecewise smooth curve with exterior angles θ_i $(1 \leq i \leq n)$, then*

$$L'(rt) = 2\pi(2 - 2h(r) - c(r)) - \int_{\Omega(r)} K + \sum_{\theta_i<0} (2\tan(\theta_i/2) - \theta_i),$$

where $h(r)$ is the number of handles in $\Omega(r)$, $c(r)$ is the number of connected components of $\Gamma(r)$, and $K(x)$ is the curvature of M at x.

Proof. Let $\Gamma(r)$ consist of smooth curves C_i $(1 \leq i \leq n)$ with endpoints x_{i-1} and x_i (where $x_0 = x_n$). Let C_i' be the arc obtained by moving each point of C_i out perpendicularly from C_i through a distance ε. Then $\Gamma(r+\varepsilon)$ coincides with $\bigcup C_i'$ except near the vertices. At each vertex x_i with a positive exterior angle θ_i, $\Gamma(r+\varepsilon)$ has an extra circular arc of length (to first order) $\varepsilon\theta_i$. At each vertex x_i with a negative exterior angle θ_i, $\bigcup C_i'$ has two extra little arcs that jut into $\Omega(r)$; to first order their length is $|2\varepsilon\tan(\theta_i/2)|$ (draw a diagram). Thus

$$\begin{aligned} L(r+\varepsilon) &= \sum |C_i'| + \sum_{\theta_i>0} \varepsilon\theta_i + \sum_{\theta_i<0} 2\varepsilon\tan(\theta_i/2) + o(\varepsilon) \\ &= L(r) + \sum \varepsilon \int_{C_i} \kappa + \sum_{\theta_i>0} \varepsilon\theta_i + \sum_{\theta_i<0} 2\varepsilon\tan(\theta_i/2) + o(\varepsilon) \end{aligned}$$

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by the first variation formula for arclength [1], where $\kappa(x)$ is the geodesic curvature of C_i at x. Hence

$$\begin{aligned} L'(r) &= \sum \int_{C_i} \kappa + \sum_{\theta_i > 0} \theta_i + \sum_{\theta_i < 0} 2\tan(\theta_i/2) \\ &= \sum \int_{C_i} \kappa + \sum \theta_i + \sum_{\theta_i < 0} (2\tan(\theta_i/2) - \theta_i) \\ &= 2\pi(2 - 2h(r) - c(r)) - \int_{\Omega(r)} K + \sum_{\theta_i < 0} (2\tan(\theta_i/2) - \theta_i) \end{aligned}$$

by the Gauss-Bonnet theorem. q.e.d.

Because $\tan\alpha < \alpha$ for $-\pi/2 < \alpha < 0$, we have

Corollary. *Under the hypotheses of the proposition,*

$$L'(r) \leq 2\pi(2 - 2h(r) - c(r)) - \int_{\Omega(r)} K.$$

References

[1] J. Cheeger & D. Ebin, *Comparison theorem in riemannian geometry*, North-Holland Math. Library, Amsterdam, 1975.

[2] B. White, *Complete surfaces of finite total curvature*, J. Differential Geometry **26** (1987) 315–326.

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